Power System Protection and Control
ISSN:1674-3415

OBSERVATIONS ON TERNARY QUADRATIC DIOPHANTINE EQUATION

$$7x^2 - 3y^2 = z^2$$

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Recevied:29 July 2024 **Revised**: 15 August 2024 **Accepted**: 25 September 2024

ABSTRACT

The homogeneous cone represented by the ternary quadratic Diophantine equation $7x^2 - 3y^2 = z^2$ is studied for finding its non – zero distinct integer solutions.

keywords: Homogeneous, Ternary quadratic equation, Integral solutions .2010 mathematics subject classification: 11D09

INTRODUCTION

While searching for problems on ternary quadratic equations, the authors came across the paper entitled [1] "On ternary quadratic Diophantine equation $7x^2 - 3y^2 = z^2$". It is noted that in [1] the author has given only a few integer solutions. This motivated us for searching other choices of integer solutions to the above equation. Also, the formulae for generating sequence of integer solutions based on a given solution are presented.

METHOD OF ANALYSIS

The ternary quadratic Diophantine equation to be solved is

$$7x^2 - 3y^2 = z^2 \qquad (1)$$

Different sets of solutions in integers to (1) are illustrated below:

PATTERN –1 :

Assume $x = x(a,b) = a^2 + 3b^2$, where $a,b > 0$ (2)

It is obvious that $7 = (2 + i\sqrt{3})(2 - i\sqrt{3})$ (3)

Substituting (2) & (3) in (1) and applying the method of factorization, one gets $z + i\sqrt{3}y = (2 + i\sqrt{3})(a + i\sqrt{3}b)^2$ (4)

Equating the real and imaginary parts, we have

$$y = y(a,b) = a^2 - 3b^2 + 4ab$$
$$z = z(a,b) = 2a^2 - 6b^2 - 6ab \qquad (5)$$

Thus (2) and (5) represent non-zero distinct integral solutions of (1) in two parameters.

PATTERN-2:

Instead of (3), Write 7 as $7 = \frac{1}{4}\left[(1 + i3\sqrt{3})(1 - i3\sqrt{3})\right]$ (6)

Following the procedure similar to pattern-1, it is found to be

$$y = y(a,b) = \frac{1}{2}[3a^2 - 9b^2 + 2ab]$$
$$z = z(a,b) = \frac{1}{2}[a^2 - 3b^2 - 18ab] \qquad (7)$$

DOI: 10.46121/pspc.52.3.2

$$x = x(A,B) = 4A^2 + 12B^2$$
$$y = y(A,B) = 6A^2 - 18B^2 + 4AB$$
$$z = z(A,B) = 2A^2 - 6B^2 - 36AB$$

Case:2

Let $a = 2A+1, b = 2B+1$

The corresponding non-zero distinct integral solutions of (9.1) are found to be

$$x = x(A,B) = 4A^2 + 12B^2 + 4A + 12B + 4$$
$$y = y(A,B) = 6A^2 - 18B^2 + 8A - 16B + 4AB - 2$$
$$z = z(A,B) = 2A^2 - 6B^2 - 16A - 24B - 36AB - 10$$

PATTERN-3:

In addition to (3) and (6), Write 7 as $7 = \frac{1}{4}\left[(5+i\sqrt{3})(5-i\sqrt{3})\right]$ (8)

Following the procedure similar to pattern-1, it is observed that

$$y = y(a,b) = \frac{1}{2}[a^2 - 3b^2 + 10ab]$$
$$z = z(a,b) = \frac{1}{2}[5a^2 - 15b^2 - 6ab] \quad (9)$$

Case:1

Let $a = 2A, b = 2B$

Thus, the corresponding non-zero distinct integral solutions of (1) are given by

$$x = x(A,B) = 4A^2 + 12B^2$$
$$y = y(A,B) = 2A^2 - 6B^2 + 20AB$$
$$z = z(A,B) = 10A^2 - 30B^2 - 12AB$$
$$x = x(p,q) = 7p^2 + q^2 + 4pq$$
$$z = z(p,q) = 14p^2 + 2q^2 + 14pq \quad (13)$$

Thus (13) and (10) represents non-zero distinct integral solutions of (1) in two parameters.

In addition to (11), 3 may also be expressed in two different ways that are presented below along with their solutions to (1) :

Way 1:

$$3 = (2\sqrt{7}+5)(2\sqrt{7}-5)$$
$$x = x(p,q) = 14p^2 + 2q^2 + 10pq$$
$$y = y(p,q) = 7p^2 - q^2$$
$$z = z(p,q) = 35p^2 + 5q^2 + 28pq$$

Way 2:

$$3 = \frac{1}{9}(2\sqrt{7}+1)(2\sqrt{7}-1)$$
$$x = x(p,q) = 42p^2 + 6q^2 + 6pq$$
$$y = y(p,q) = 63p^2 - 9q^2$$
$$z = z(p,q) = 21p^2 + 3q^2 + 84pq$$

PATTERN-5

Introduction of the linear transformations

$$x = u \pm 3v, y = u \pm 7v, z = 2w \quad (14)$$

in (1) leads to

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$$w^2 + 21v^2 = u^2 \qquad (15)$$

Write (15) as the system of double equations as shown in Table 1 below:

Table 1: System of double equations

System	1	2	3	4	5
$u+w$	$21v$	$7v$	v^2	$7v^2$	$3v^2$
u-w	v	$3v$	21	3	7

Solving each of the system of equations in Table 1, the corresponding values of u,w and v are obtained. Substituting the values of u,w and v in (14), the respective values of x and y are determined. For simplicity and brevity, the integer solutions to (1) obtained through solving each of the above system of equations are exhibited.

System :1

$$x = 14T$$
$$y = 18T$$
$$z = 20T$$

1System:2

$$x = 8T$$
$$y = 12T$$
$$z = 4T$$

System:3

$$x = 2T^2 + 8T + 14$$
$$y = 2T^2 + 16T + 18$$
$$z = 4T^2 + 4T - 20$$

System :4

$$x = 14T^2 + 20T + 8$$
$$y = 14T^2 + 28T + 12$$
$$z = 28T^2 + 28T + 4$$

System:5

$$x = 6T^2 + 12T + 8$$
$$y = 6T^2 + 20T + 12$$
$$z = 12T^2 + 12T - 4$$

PATTERN-6

Rewrite (15) as $w^2 + 21v^2 = u^2 * 1$ (16)

Assume $u = u(m,n) = m^2 + 21n^2$, where $m,n > 0$ (17)

It is noted that $1 = \dfrac{(21r^2 - s^2 + i\sqrt{21}rs)(21r^2 - s^2 - i\sqrt{21}rs)}{(21r^2 + s^2)^2}$ (18)

As our interest centers on finding integer solutions, it seen that w and v are integers, when $m = (21r^2 + s^2)M$, $n = (21r^2 + s^2)N$

In view of (14), the corresponding non-zero distinct integral solutions of (1) are given by

$$x = (21r^2 + s^2)(M^2 + 21N^2) \pm 3(21r^2 + s^2)\left[2MN(21r^2 - s^2) + 2rs(21r^2 + s^2)(M^2 - 21N^2)\right]$$
$$y = (21r^2 + s^2)(M^2 + 21N^2) \pm 7(21r^2 + s^2)\left[2MN(21r^2 - s^2) + 2rs(21r^2 + s^2)(M^2 - 21N^2)\right]$$
$$z = 2(21r^2 + s^2)\left[(21r^4 - s^4)(M^2 - 21N^2) - 84rs(21r^2 + s^2)MN\right]$$

REMARKABLE OBSERVATIONS

- If the non-zero integer triplet (x_0, y_0, z_0) is any solution of (1) then each of the following three triplets of integer based on x_0, y_0 and z_0 also satisfies (1).

Triplet:1 (x_n, y_n, z_0)

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$$x_n = \frac{1}{2\sqrt{21}}\left[3B_n y_0 + \sqrt{21}A_n x_0\right]$$

$$y_n = \frac{1}{2\sqrt{21}}\left[\sqrt{21}A_n y_0 + 7B_n x_0\right],$$

$$z_n = z_0$$

where

$$A_n = (55+12\sqrt{21})^n + (55-12\sqrt{21})^n$$

$$B_n = (55+12\sqrt{21})^n - (55-12\sqrt{21})^n$$

Triplet:2 (x_n, y_0, z_n)

$$x_n = \frac{1}{2\sqrt{7}}\left[B_n z_0 + \sqrt{7}A_n x_0\right]$$

$$y_n = y_0 \quad ,$$

$$z_n = \frac{1}{2\sqrt{7}}\left[\sqrt{7}A_n z_0 + 7B_n x_0\right]$$

where

$$A_n = (8+3\sqrt{7})^n + (8-3\sqrt{7})^n$$

$$B_n = (8+3\sqrt{7})^n - (8-3\sqrt{7})^n$$

CONCLUSION

To conclude, one may search for other patterns of solutions and their corresponding properties.

REFERENCE

[1]. R.Nandhini, On Ternary Quadratic equation, Paripex- Indian Journal of Research, Vol 5(3), (2016),66-68