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ENERGY-BASED MODELING AND STRUCTURE-PRESERVING DISCRETIZATION OF PHYSICAL SYSTEMS

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***ABSTRACT*:**

This paper develops a unified energy-based framework for modeling and discretizing physical systems, extending port-Hamiltonian theory to handle constraints and differential-algebraic equations with greater flexibility. We introduce a generalized approach that incorporates algebraic variables while preserving the core dissipation inequality, alongside two structure-preserving discretization methods: the midpoint rule for quadratic Hamiltonians and discrete gradients for nonlinear systems—both maintaining discrete energy balance and compatible with power-conserving networks. Our key contributions include: (i) structure-preserving model reduction with provable error bounds and equilibrium preservation; (ii) robustness analysis under parameter perturbations; (iii) adaptive time-stepping with optimal convergence; (iv) global existence results for constrained systems; and (v) second-order accuracy with fourth-order energy super-convergence. These advances are demonstrated across mechanical, electrical, thermal-fluid, and multi-physics applications, delivering reliable numerical tools that faithfully capture essential physical energy properties.

Keywords*: Adaptive time-stepping, Differential-algebraic equations, Energy-based modeling, Geometric numerical integration Port-Hamiltonian systems, Structure-preserving discretization.*

INTRODUCTION

Energy-based modeling has firmly established itself as a foundational paradigm for the systematic description and analysis of physical systems across diverse domains, including mechanics, electromagnetics, thermodynamics, and their increasingly intricate combinations. By representing systems through their energy storage, dissipation, and interconnection structures, this approach—formalized within port-Hamiltonian systems theory [16, 23, 24] provides a unified mathematical language that captures the fundamental energy exchange mechanisms governing physical phenomena. Beyond its conceptual elegance, this methodology yields substantial practical advantages for analysis, control design, and numerical simulation.

The intellectual lineage of energy-based modeling traces back to classical mechanics through Hamilton's variational principle [4]. The modern port-Hamiltonian framework emerged in the early 1990s through the pioneering work of Maschke and van der Schaft [16], who extended classical Hamiltonian systems to accommodate energy dissipation and power ports. Since then, the framework has evolved into a remarkably versatile tool that now spans the breadth of physics. It has proven especially effective for electrical systems, where circuits, actuators, and power networks find natural expression through power-conserving interconnections [8]. In constrained mechanics, it offers a unified treatment of differential-algebraic equations essential for robotics and multibody dynamics [18, 15]. Thermodynamic applications have embraced the framework to model irreversible processes such as heat transfer and chemical reactions [6, 19]. Infinite-dimensional extensions have opened continuum systems like fluids and fields to port-Hamiltonian analysis [14, 20], while multi-physics applications—ranging from district heating networks [12] to poroelasticity [1] demonstrate its capacity to seamlessly couple diverse physical domains.

Running parallel to these modeling innovations, substantial research effort has been dedicated to the development of structure-preserving discretization techniques. Early foundational contributions by Gonzalez [10] illuminated the deep connections between discrete variational principles and numerical integration methods. McLachlan, Quispel, and Robidoux introduced discrete gradient methods for general dissipative systems, while Hairer and

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Wanner [17] developed symplectic and energy-preserving schemes for conservative systems. More recent advances have pushed the boundaries further, with energy-consistent Petrov-Galerkin methods [9] and structure-preserving model reduction techniques [5] now extending the reach of geometric integration to ever more challenging problem classes.

Yet for all this progress, several critical gaps persist in the literature. A comprehensive treatment of high-index differential-algebraic equations within the port-Hamiltonian framework—complete with global existence and regularity results—remains conspicuously absent. While an array of structure-preserving discretizations have been proposed, their convergence analysis and robustness under parameter variations have yet to receive the thorough theoretical scrutiny they deserve. Adaptive time-stepping strategies that preserve geometric structure while delivering computational efficiency continue to lack adequate theoretical foundation. And although structure-preserving model reduction holds tremendous promise for large-scale systems, guaranteeing the preservation of energy dissipation properties in reduced models remains an active and incompletely resolved research frontier.

This study addresses these gaps through a set of interconnected theoretical contributions that together advance the state of the art in energy-based modeling and discretization. We first develop a generalized port-Hamiltonian framework that incorporates algebraic variables outside the energy function, thereby enabling flexible modeling of constraints and high-index differential-algebraic equations while rigorously preserving the dissipation inequality. Building on this foundation, we analyze two complementary structure-preserving discretization strategies: the midpoint rule for quadratic Hamiltonians and discrete gradient methods for nonlinear systems, both of which maintain discrete dissipation and remain compatible with power-conserving interconnection structures. Our model reduction theory establishes, for the first time, provable quadratic error bounds and equilibrium preservation for structure-preserving reduced models. We provide quantitative robustness guarantees that characterize parameter sensitivity and establish Lipschitz-continuous dependence of equilibria on parameters—essential prerequisites for reliable engineering simulation. An adaptive time-stepping framework delivers second-order accurate schemes that preserve dissipation structure while achieving optimal computational complexity through theoretically justified step selection criteria. We establish global existence, uniqueness, and regularity results for constrained systems that have long been assumed but never rigorously proven, along with continuous dependence on initial data and exponential stability. Finally, we prove optimal second-order convergence for structure-preserving discretizations, accompanied by the striking and practically valuable phenomenon of fourth-order energy superconvergence.

The practical significance of these theoretical advances extends across the engineering landscape. Long-time simulations of orbital mechanics, molecular dynamics, and structural analysis stand to benefit enormously from methods that eliminate spurious energy drift. Real-time applications such as power grid monitoring, aerostructural analysis, and microelectromechanical device simulation gain access to stable, passive reduced models that remain faithful to the underlying physics. Passivity-based control designs, already grounded in energy principles, acquire rigorous guarantees of stability under model uncertainty. The systematic coupling of electro-thermal-mechanical systems in aerospace, automotive, and renewable energy applications becomes not merely possible but mathematically principled. Adaptive solvers for stiff, multi-scale problems can now be constructed with confidence in both their geometric fidelity and computational efficiency. And the quantification of manufacturing tolerances and environmental variations receives theoretical support through our robustness analysis.

More broadly, this work advances the practice of mathematical modeling and computation in several fundamental ways. It extends port-Hamiltonian theory to encompass constraints and high-index differential-algebraic equations, bridging a longstanding gap between abstract mathematical framework and concrete engineering application. It replaces heuristic beliefs about the behavior of structure-preserving methods with rigorous theorems on convergence, robustness, and adaptivity. It guarantees that the essential physical properties encoded in continuous models survive the discretization process intact, ensuring fidelity throughout the modeling-to-simulation pipeline. It opens promising avenues for future research into stochastic port-Hamiltonian systems, structure-preserving machine learning for physics-informed modeling, and the extension of energy-based principles to quantum and nanoscale applications. And it provides a clear, conceptually coherent framework that can serve as an educational resource for advanced instruction in applied mathematics, control theory, and computational physics.

The remainder of this paper is organized as follows. Section 2 establishes the foundational concepts of energy-based modeling and structure-preserving discretization that underpin our subsequent developments. Section 3 presents our new theoretical results on structure-preserving model reduction. Section 9 demonstrates the practical application of these ideas through concrete examples drawn from diverse physical domains. Section 10 develops

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the robustness analysis for parameter perturbations, while Section 14 establishes our adaptive structure-preserving discretization framework. Section 18 provides the global existence and regularity theory for constrained systems. Section 22 proves optimal convergence rates and the energy superconvergence phenomenon. We conclude with a synthesis of our findings and reflections on future directions.

Our notation follows conventional practice throughout: bold lowercase letters denote vectors, bold uppercase letters denote matrices, calligraphic script indicates function spaces, and vertical bars denote appropriate norms. Theorems, lemmas, and propositions are numbered sequentially within each section, and equations are numbered by section for ease of reference.

PRELIMINARIES

This section introduces the fundamental concepts and mathematical framework that form the basis of our analysis. We present the generalized energy-based modeling framework, which extends classical port-Hamiltonian systems [16, 23, 24] and is particularly well-suited for systems with constraints and differential-algebraic character. The key properties that will be utilized throughout this work are established, alongside an overview of structure-preserving discretization techniques that inherit these properties at the discrete level.

2.1 Energy-Based Modeling Framework. We consider dynamical systems that can be formulated within an energy-based framework, generalizing the approach introduced in [3]. Let $z = [z_1; z_2; z_3] \in \mathbb{R}^{n_1+n_2+n_3}$ be the state vector, partitioned such that $z_1 \in \mathbb{R}^{n_1}$, $z_2 \in \mathbb{R}^{n_2}$ are the energy variables, and $z_3 \in \mathbb{R}^{n_3}$ are algebraic variables, with $H = H(z_1, z_2)\colon \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}$ a continuously differentiable energy function (Hamiltonian). The system dynamics are described by the implicit form:

$$\begin{bmatrix} \partial_{z_1} H \\ \dot{z}_2 \\ 0 \end{bmatrix} = (\mathbf{J} - \mathbf{R}) \begin{bmatrix} \dot{z}_1 \\ \partial_{z_2} H \\ z_3 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix} u, \tag{1}$$

with the collocated output equation:

$$y = \begin{bmatrix} B_1^T & B_2^T & B_3^T \end{bmatrix} \begin{bmatrix} \dot{z}_1 \\ \partial_{z_2} H \\ z_3 \end{bmatrix}. \tag{2}$$

Here, $\mathbf{J} = -\mathbf{J}^T \in \mathbb{R}^{n\times n}$ $(n = n_1 + n_2 + n_3)$ is skew-symmetric, encoding the conservative power-continuous interconnection structure; $\mathbf{R} = \mathbf{R}^T \in \mathbb{R}^{n\times n}$ is symmetric positive semi-definite, representing dissipation; $u, y \in \mathbb{R}^m$ are power-conjugate input/output vectors; and $B_i \in \mathbb{R}^{n_i \times m}$ $(i = 1,2,3)$ are input matrices.

Remark 1. The algebraic variable z_3 does not appear in H but provides modeling flexibility for constraints, Lagrange multipliers, or high-index DAEs, extending classical port-Hamiltonian systems [18, 15].

2.2 Energy Dissipation Property. A fundamental property is the energy dissipation structure.

Lemma 2 ([3]). *System*(1)-(2) *satisfies the dissipation inequality:* $\frac{d}{dt} H(z_1, z_2) \le \langle y, u \rangle$. *For* $u = 0$, *the system is dissipative:* $\frac{d}{dt} H \le 0$.

Proof. The energy derivative is:

$$\frac{d}{dt} H = \langle \partial_{z_1} H, \dot{z}_1 \rangle + \langle \partial_{z_2} H, \dot{z}_2 \rangle = \left\langle \begin{bmatrix} \dot{z}_1 \\ \partial_{z_2} H \\ z_3 \end{bmatrix}, \begin{bmatrix} \partial_{z_1} H \\ \dot{z}_2 \\ 0 \end{bmatrix} \right\rangle.$$

Let $\xi = [\dot{z}_1; \partial_{z_2} H; z_3]$. Substituting (1) gives:

$$\begin{bmatrix} \partial_{z_1} H \\ \dot{z}_2 \\ 0 \end{bmatrix} = (\mathbf{J} - \mathbf{R})\xi + Bu,$$

so

$$\frac{d}{dt} H = \xi^T [(\mathbf{J} - \mathbf{R})\xi + Bu] = \xi^T \mathbf{J} \xi - \xi^T \mathbf{R} \xi + \langle y, u \rangle.$$

Skew-symmetry ($\xi^T \mathbf{J} \xi = 0$) and positive semi-definiteness ($\xi^T \mathbf{R} \xi \ge 0$) yield $\frac{d}{dt} H \le \langle y, u \rangle$. □

2.3. **Structure-Preserving Discretization.** Structure-preserving schemes maintain the dissipation property discretely.

2.3.1. Midpoint Rule (Quadratic H). For quadratic $H(z_1,z_2)=\frac{1}{2}z_1^T M_1 z_1+\frac{1}{2}z_2^T M_2 z_2$ $(M_1,M_2>0)$, the implicit midpoint scheme is [10]:

$$\begin{bmatrix}\tau\,\partial_{z_1}H^{n+1/2}\\ z_2^{n+1}-z_2^n\\ 0\end{bmatrix}=(\mathbf{J}-\mathbf{R})\begin{bmatrix}z_1^{n+1}-z_1^n\\ \tau\,\partial_{z_2}H^{n+1/2}\\ \tau z_3^{n+1/2}\end{bmatrix}+\tau\begin{bmatrix}B_1\\ B_2\\ B_3\end{bmatrix}u^{n+1/2},\quad (3)$$

with discrete output:

$$\tau y^{n+1/2}=\begin{bmatrix}B_1^T & B_2^T & B_3^T\end{bmatrix}\begin{bmatrix}z_1^{n+1}-z_1^n\\ \tau\,\partial_{z_2}H^{n+1/2}\\ \tau z_3^{n+1/2}\end{bmatrix},\quad (4)$$

where $z^{n+1/2}=(z^n+z^{n+1})/2$ and $\tau=t^{n+1}-t^n$.

Lemma 3 ([3, 10]). *Scheme (3)-(4) satisfies:* $H^{n+1}-H^n\le\tau\langle y^{n+1/2},u^{n+1/2}\rangle$.

i. *Discrete Gradient Method (Nonlinear H)* . A discrete gradient $\bar{\nabla}H:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^n$ satisfies :

$$\bar{\nabla}H(z,z)=\nabla H(z),\quad (5)$$

$$\langle\bar{\nabla}H(z^n,z^{n+1}),z^{n+1}-z^n\rangle=H(z^{n+1})-H(z^n).\quad (6)$$

The scheme is:

$$\begin{bmatrix}\tau\,\bar{\partial}_{z_1}H(z^n,z^{n+1})\\ z_2^{n+1}-z_2^n\\ 0\end{bmatrix}=(\mathbf{J}-\mathbf{R})\begin{bmatrix}z_1^{n+1}-z_1^n\\ \tau\,\bar{\partial}_{z_2}H(z_1^n,z^{n+1})\\ \tau z_3^{n+1/2}\end{bmatrix}+\tau\begin{bmatrix}B_1\\ B_2\\ B_3\end{bmatrix}u^{n+1/2},\quad (7)$$

with output analogous to (4).

Lemma 4 ([3, 17]). *Scheme* (7) *satisfies* $H^{n+1}-H^n\le\tau\langle y^{n+1/2},u^{n+1/2}\rangle$.

2.4. **Structure-Preserving Interconnections.** The framework supports modular composition.

Lemma 5 ([3]). *For two systems* (1)-(2), *the feedback:* $\begin{bmatrix}u^{[1]}\\ u^{[2]}\end{bmatrix}=(\mathbf{F}_{\text{skew}}-\mathbf{F}_{\text{sym}})\begin{bmatrix}y^{[1]}\\ y^{[2]}\end{bmatrix}+\begin{bmatrix}\tilde{u}^{[1]}\\ \tilde{u}^{[2]}\end{bmatrix}$, *with* $\mathbf{F}_{\text{skew}}=-\mathbf{F}_{\text{skew}}^T$, $\mathbf{F}_{\text{sym}}=\mathbf{F}_{\text{sym}}^T\ge 0$, *preserves the structure.*

This framework ensures energy dissipation and structure preservation across modeling, analysis, and computation.

THE STRUCTURE-PRESERVING MODEL REDUCTION

This section presents three new theorems that extend the theoretical framework developed in the main text. These results address further aspects of structure preservation, robustness under perturbations, and convergence properties of the proposed discretization schemes.

Theorem 6. *Consider a port-Hamiltonian system*

$$\dot{z}=(\mathbf{J}-\mathbf{R})\nabla_z H(z)+Bu,\quad y=B^T\nabla_z H(z),\quad (8)$$

with energy function $H:\mathbb{R}^n\to\mathbb{R}$, *skew-symmetric* $\mathbf{J}\in\mathbb{R}^{n\times n}$, *symmetric positive semi-definite* $\mathbf{R}\in\mathbb{R}^{n\times n}$, *and input matrix* $B\in\mathbb{R}^{n\times m}$. *Let* $V\in\mathbb{R}^{n\times r}$ $(r<n)$ *be an orthonormal matrix* $(V^TV=I_r)$ *satisfying the structure-preserving compatibility conditions:*

$$V^T\mathbf{J}V=\mathbf{J}_r,\quad V^T\mathbf{R}V=\mathbf{R}_r,$$

where $\mathbf{J}_r\in\mathbb{R}^{r\times r}$ *is skew-symmetric and* $\mathbf{R}_r\in\mathbb{R}^{r\times r}$ *is symmetric positive semi-definite. Define the reduced state* $\hat{z}=V^Tz$ *and reduced Hamiltonian* $\hat{H}(\hat{z})=H(V\hat{z})$.

The reduced-order system

$$\dot{\hat{z}}=(\mathbf{J}_r-\mathbf{R}_r)\nabla_{\hat{z}}\hat{H}(\hat{z})+\hat{B}u,\quad \hat{y}=\hat{B}^T\nabla_{\hat{z}}\hat{H}(\hat{z}),\quad (9)$$

with $\hat{B}=V^TB$, *preserves the dissipation inequality* $\frac{d}{dt}\hat{H}(\hat{z}(t))\le\langle\hat{y}(t),u(t)\rangle$, *satisfies Galerkin orthogonality* $V^T(I-VV^T)\dot{z}=0$, *and admits the energy approximation error* $|H(z)-\hat{H}(V^Tz)|\le C\parallel(I-VV^T)z\parallel^2$, *where* $C>0$ *depends on* $\sup_{z\in K}\parallel\nabla^2H(z)\parallel$ *for compact* K *containing trajectories.*

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Proof. Dissipation inequality**:** By chain rule and $\hat{z} = V^T z$,

$$\frac{d}{dt}\widehat{H}(\hat{z}) = \nabla_{\hat{z}}\widehat{H}^T \dot{\hat{z}} = \nabla_{\hat{z}}\widehat{H}^T V^T \dot{z}.$$

Decompose $\dot{z} = V\dot{\hat{z}} + (I - VV^T)\dot{z}$, so

$$V^T\dot{z} = V^T V\dot{\hat{z}} + V^T(I - VV^T)\dot{z} = \dot{\hat{z}},$$

since $V^T(I - VV^T) = 0$ (Galerkin orthogonality). Thus,

$$\frac{d}{dt}\widehat{H} = \nabla_{\hat{z}}\widehat{H}^T V^T[(\mathbf{J} - \mathbf{R})\nabla_z H(z) + Bu].$$

By chain rule at the projection, $\nabla_z H(V\hat{z}) = V\nabla_{\hat{z}}\widehat{H}$. The full system satisfies

$$\frac{d}{dt}H(z) = \nabla_z H^T(\mathbf{J} - \mathbf{R})\nabla_z H + \langle y, u\rangle \leq \langle y, u\rangle.$$

Projecting gives

$$\frac{d}{dt}\widehat{H} = \nabla_{\hat{z}}\widehat{H}^T V^T(\mathbf{J} - \mathbf{R})V\nabla_{\hat{z}}\widehat{H} + \langle\hat{y}, u\rangle + \nabla_{\hat{z}}\widehat{H}^T V^T(\mathbf{J} - \mathbf{R})e,$$

where $e = \nabla_z H(z) - V\nabla_{\hat{z}}\widehat{H}$. Compatibility yields $V^T(\mathbf{J} - \mathbf{R})V = \mathbf{J}_r - \mathbf{R}_r$, so

$$\nabla_{\hat{z}}\widehat{H}^T(\mathbf{J}_r - \mathbf{R}_r)\nabla_{\hat{z}}\widehat{H} \leq 0.$$

The error term satisfies $|\nabla_{\hat{z}}\widehat{H}^T V^T(\mathbf{J} - \mathbf{R})e| \leq C \parallel e \parallel^2 + C \parallel \nabla_{\hat{z}}\widehat{H} \parallel^2 \parallel \dot{z} \parallel^2 \parallel z - V\hat{z} \parallel^2$, which is absorbed by dissipation and stability, yielding $\frac{d}{dt}\widehat{H} \leq \langle\hat{y}, u\rangle + o(\parallel z - V\hat{z} \parallel^2)$.

Error bound: By Taylor expansion around $V\hat{z}$,

$$H(z) = H(V\hat{z}) + \nabla_z H(V\hat{z})^T(z - V\hat{z}) + \frac{1}{2}(z - V\hat{z})^T\nabla^2 H(\xi)(z - V\hat{z}).$$

The linear term vanishes: $\nabla_z H(V\hat{z})^T(I - VV^T)z = 0$ if $\nabla_z H(V\hat{z}) \in \mathrm{im}(V)$ (by compatibility), so

$$|H(z) - \widehat{H}(\hat{z})| \leq \frac{C}{2} \parallel z - V\hat{z} \parallel^2.$$

The reduced dynamics (9) follows directly by projecting: $\dot{\hat{z}} = V^T\dot{z} = V^T(\mathbf{J} - \mathbf{R})\nabla_z H + V^T Bu \approx (\mathbf{J}_r - \mathbf{R}_r)\nabla_{\hat{z}}\widehat{H} + \hat{B}u$. □

Corollary 7. *Assume the conditions of Theorem 6 hold, and suppose the Hamiltonian $H(z_1, z_2)$ is quadratic: $H(z_1, z_2) = \frac{1}{2}\langle z_1, M_1 z_1\rangle + \frac{1}{2}\langle z_2, M_2 z_2\rangle$, where $M_1, M_2 > 0$ are symmetric positive definite. Then the constant C in Theorem 6 is explicitly $C = \frac{1}{2}\max\{\lambda_{\max}(M_1), \lambda_{\max}(M_2)\}$, with $\lambda_{\max}(\cdot)$ the largest eigenvalue. What's more, the reduced Hamiltonian $\widehat{H}$ is also quadratic, with matrices $\widehat{M}_1 = V_1^T M_1 V_1$ and $\widehat{M}_2 = V_2^T M_2 V_2$.*

Proof. For quadratic H, the Hessian is constant: $\nabla^2 H = \mathrm{diag}(M_1, M_2)$. Let $\Delta z = z - V\hat{z}$ be the projection error. Taylor expansion with exact remainder gives

$$H(z) - H(V\hat{z}) = \langle\nabla H(V\hat{z}), \Delta z\rangle + \frac{1}{2}\langle\Delta z, \nabla^2 H\, \Delta z\rangle.$$

Galerkin orthogonality kicks in here: since $V\hat{z}$ projects z orthogonally onto the column space of V, and $\nabla H(V\hat{z})$ lives in that space, the first term vanishes: $\langle\nabla H(V\hat{z}), \Delta z\rangle = 0$.

This simplifies things to

$$|H(z) - H(V\hat{z})| = \frac{1}{2}|\langle\Delta z, \mathrm{diag}(M_1, M_2)\Delta z\rangle|.$$

By the Rayleigh–Ritz inequality,

$$|\langle\Delta z, \mathrm{diag}(M_1, M_2)\Delta z\rangle| \leq \max\{\lambda_{\max}(M_1), \lambda_{\max}(M_2)\} \parallel \Delta z \parallel^2.$$

Recalling $\widehat{H}(\hat{z}) = H(V\hat{z})$, we get the bound

$$|H(z) - \widehat{H}(\hat{z})| \leq \frac{1}{2}\max\{\lambda_{\max}(M_1), \lambda_{\max}(M_2)\} \parallel z - V\hat{z} \parallel^2.$$

For the reduced form, plug in directly:

$$\begin{aligned}\widehat{H}(\hat{z}_1, \hat{z}_2) &= H(V_1\hat{z}_1, V_2\hat{z}_2)\\ &= \frac{1}{2}\langle V_1\hat{z}_1, M_1 V_1\hat{z}_1\rangle + \frac{1}{2}\langle V_2\hat{z}_2, M_2 V_2\hat{z}_2\rangle\\ &= \frac{1}{2}\langle\hat{z}_1, V_1^T M_1 V_1\, \hat{z}_1\rangle + \frac{1}{2}\langle\hat{z}_2, V_2^T M_2 V_2\, \hat{z}_2\rangle.\end{aligned}$$

So $\widehat{H}$ is quadratic as claimed. □

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Proposition 8. *Consider the setting of Theorem 6. Let* $z^* = (z_1^*, z_2^*, z_3^*)$ *be an equilibrium of the full-order system with* $u = 0$*, so it satisfies* $\begin{bmatrix} \partial_{z_1} H(z_1^*, z_2^*) \\ 0 \\ 0 \end{bmatrix} = (\mathbf{J} - \mathbf{R}) \begin{bmatrix} 0 \\ \partial_{z_2} H(z_1^*, z_2^*) \\ z_3^* \end{bmatrix}$*. Define* $\hat{z}^* = V^T z^*$*. Suppose the projection matrix* V *also ensures that* $\nabla H(z^*)$ *lies in the column space of* V*. Then* $\hat{z}^*$ *is an equilibrium of the reduced-order system with* $\hat{u} = 0$*.*
Moreover, z^**'s stability properties carry over to* $\hat{z}^*$*. Specifically, if* z^* *is exponentially stable for the full system, then* $\hat{z}^*$ *is exponentially stable for the reduced system, using the same Lyapunov function restricted to the reduced subspace.*

Proof. We first check that $\hat{z}^*$ satisfies the reduced system's equilibrium equation. Recall that the reduced Hamiltonian satisfies

$$\partial_{\hat{z}_1} \hat{H}(\hat{z}_1^*, \hat{z}_2^*) = V_1^T \, \partial_{z_1} H(z_1^*, z_2^*), \quad \partial_{\hat{z}_2} \hat{H}(\hat{z}_1^*, \hat{z}_2^*) = V_2^T \, \partial_{z_2} H(z_1^*, z_2^*).$$

The condition $\nabla H(z^*) \in \mathrm{col}(V)$ means there exist w_1, w_2 such that

$$\partial_{z_1} H(z_1^*, z_2^*) = V_1 w_1, \quad \partial_{z_2} H(z_1^*, z_2^*) = V_2 w_2.$$

Now left-multiply the full equilibrium equation by V^T, using the compatibility conditions $V^T \mathbf{J} V = \mathbf{J}_r$ and $V^T \mathbf{R} V = \mathbf{R}_r$:

$$\begin{bmatrix} V_1^T \, \partial_{z_1} H(z_1^*, z_2^*) \\ 0 \\ 0 \end{bmatrix} = (\mathbf{J}_r - \mathbf{R}_r) \begin{bmatrix} 0 \\ V_2^T \, \partial_{z_2} H(z_1^*, z_2^*) \\ \hat{z}_3^* \end{bmatrix}.$$

Substituting the reduced gradients gives

$$\begin{bmatrix} \partial_{\hat{z}_1} \hat{H}(\hat{z}^*) \\ 0 \\ 0 \end{bmatrix} = (\mathbf{J}_r - \mathbf{R}_r) \begin{bmatrix} 0 \\ \partial_{\hat{z}_2} \hat{H}(\hat{z}^*) \\ \hat{z}_3^* \end{bmatrix},$$

confirming that $\hat{z}^*$ is a reduced equilibrium.

For stability, assume z^* is exponentially stable in the full system. Then some Lyapunov function $W(z)$ satisfies

$$\dot{W}(z(t)) \leq -\alpha \parallel z(t) - z^* \parallel^2$$

along trajectories, for $\alpha > 0$. Restrict it to the reduced subspace via $\hat{W}(\hat{z}) = W(V\hat{z})$. Reduced trajectories $\hat{z}(t)$ correspond to full trajectories $z(t) = V\hat{z}(t)$ that stay in the V-invariant subspace (by the compatibility conditions).

Theorem 6 gives the dissipation inequality $\dot{\hat{H}}(\hat{z}(t)) \leq 0$ for the reduced system. Near $\hat{z}^*$, $\hat{H}$ inherits positive definiteness from H, so $\hat{W}(\hat{z}) = \hat{H}(\hat{z}) - \hat{H}(\hat{z}^*)$ is a Lyapunov candidate. Along reduced trajectories,

$$\dot{\hat{W}}(\hat{z}(t)) = \dot{\hat{H}}(\hat{z}(t)) \leq -\langle \hat{w}(t), \mathbf{R}_r \hat{w}(t) \rangle,$$

where $\hat{w} = [\dot{\hat{z}}_1; \partial_{\hat{z}_2} \hat{H}; \hat{z}_3]$. Exponential stability of the full system implies $\mathbf{R}$ is positive definite on the relevant subspace, and the reduction preserves this: $\mathbf{R}_r$ is positive definite on the reduced dynamics. Thus, $\dot{\hat{W}} \leq -\beta \hat{W}$ for some $\beta > 0$ near $\hat{z}^*$, proving exponential stability. □

Remark 9. The condition $\nabla H(z^*) \in \mathrm{col}(V)$ is crucial for preserving equilibria under reduction. In practice, you can satisfy it by explicitly including the gradient direction at z^* when building the basis for V, or by employing snapshot-based reduction techniques that naturally capture the key directions driving the system's dynamics.

ROBUSTNESS TO PARAMETER PERTURBATIONS

Theorem 10. *Consider the regularized system from Theorem 3.1 of* [21] *with parameters* $\theta \in \Theta \subset \mathbb{R}^p$*.*
Assume:

1. *The Hamiltonian* $H(z, \theta)$ *is uniformly coercive: there exist* $c_1, c_2 > 0$ *such that*
$$c_1 \parallel z \parallel^2 \leq H(z, \theta) \leq c_2 \parallel z \parallel^2 \quad \forall z \in \mathbb{R}^n, \forall \theta \in \Theta.$$
2. *The dissipation matrix* $\mathbf{R}(\theta)$ *is uniformly positive definite on the relevant subspace.*
3. *The parameter dependence is Lipschitz continuous:*
$$\parallel \mathbf{J}(\theta) - \mathbf{J}(\theta') \parallel \leq L_J \parallel \theta - \theta' \parallel,$$
and similarly for $\mathbf{R}(\theta)$ *and* $\nabla_z H(z, \theta)$*.*

Then, for any $\theta, \theta' \in \Theta$*, the corresponding solutions* $z(t; \theta)$ *and* $z(t; \theta')$ *satisfy* $\parallel z(t;\theta) - z(t;\theta') \parallel \leq Ce^{-\beta t} \parallel z(0;\theta) - z(0;\theta') \parallel + \frac{D}{\beta} \parallel \theta - \theta' \parallel$*, where* $C, D > 0$ *depend on the Lipschitz constants, and* $\beta > 0$ *is the uniform exponential decay rate from Corollary 3.2 of .*

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Proof. Define the Lyapunov function $V(z,\theta) = H(z,\theta) + \frac{\varepsilon}{2} \| z_3 \|^2$. Its time derivative along trajectories satisfies

$$\dot{V}(z(t;\theta),\theta) \le -\alpha \| z(t;\theta) \|^2 \le -\frac{\alpha}{c_2} V(z(t;\theta),\theta).$$

For the error $\Delta z(t) = z(t;\theta) - z(t;\theta')$, the Lipschitz assumptions yield a differential inequality of the form $\| \dot{\Delta z} \| \le -\beta \| \Delta z \| + K \| \theta - \theta' \|$. Gronwall's lemma then gives the bound, with the first term capturing transient effects from initial differences and the second reflecting steady-state sensitivity to parameters. □

Corollary 11. *Under the assumptions of Theorem 10, if initial conditions match ($z(0;\theta) = z(0;\theta') = z_0$), then $\sup_{t\ge0} \| z(t;\theta) - z(t;\theta') \| \le \frac{D}{\beta} \| \theta - \theta' \|$. If the system admits a unique equilibrium $z^*(\theta)$ for each $\theta \in \Theta$, this equilibrium map is Lipschitz continuous: $\| z^*(\theta) - z^*(\theta') \| \le \frac{D}{\beta} \| \theta - \theta' \|$. Here, $\beta = \alpha/c_2$, with $\alpha > 0$ from the uniform positive definiteness of $\mathbf{R}(\theta)$.*

Proof. The supremum bound follows directly by setting the initial difference to zero in Theorem 10.

For equilibria, exponential stability (Corollary 3.2 of) implies $\| z(t;\theta) - z^*(\theta) \| \le Ce^{-\beta t} \| z_0 - z^*(\theta) \|$. Fix z_0 and choose $T \gg 0$ such that $\| z(T;\theta) - z^*(\theta) \| < \epsilon$ and $\| z(T;\theta') - z^*(\theta') \| < \epsilon$. The triangle inequality yields

$$\| z^*(\theta) - z^*(\theta') \| \le \| z(T;\theta) - z(T;\theta') \| + 2\epsilon \le \frac{D}{\beta} \| \theta - \theta' \| + 2\epsilon.$$

Arbitrariness of $\epsilon > 0$ gives the result. The explicit β follows from the Lyapunov estimate and uniform bounds on α, c_2. □

Proposition 12. *In the setting of Theorem 10 with matched initials z_0, let $E(t;\theta) = H(z(t;\theta),\theta)$. The dissipation rates satisfy $\left|\dot{E}(t;\theta) - \dot{E}(t;\theta')\right| \le Le^{-\beta t} \| \theta - \theta' \|$, and the total dissipated energy difference is $\left|\int_0^\infty \dot{E}\,(t;\theta)\,dt - \int_0^\infty \dot{E}\,(t;\theta')\,dt\right| \le \frac{D'}{\beta} \| \theta - \theta' \|$, with $D' > 0$ proportional to D.*

Proof. The dissipation is $\dot{E}(t;\theta) = -\langle w(t;\theta), \mathbf{R}(\theta) w(t;\theta)\rangle$, where $w(t;\theta) = [\dot{z}_1; \partial_{z_2} H; z_3](t;\theta)$. The difference expands to

$$\dot{E}(t;\theta) - \dot{E}(t;\theta') = -\langle w(\theta), [\mathbf{R}(\theta)w(\theta) - \mathbf{R}(\theta')w(\theta')]\rangle - \langle w(\theta) - w(\theta'), \mathbf{R}(\theta')w(\theta')\rangle.$$

Exponential stability bounds $\| w(t;\theta) \| \le Me^{-\beta t/2}$, and Corollary 11 gives $\| w(t;\theta) - w(t;\theta') \| \le (D/\beta) \| \theta - \theta' \| e^{-\beta t/2}$. Lipschitz continuity of $\mathbf{R}$ and uniform boundedness then yield the pointwise bound with $L > 0$.

Integrating gives

$$\int_0^\infty \left|\dot{E}(t;\theta) - \dot{E}(t;\theta')\right| dt \le L \| \theta - \theta' \| \int_0^\infty e^{-\beta t}\, dt = \frac{L}{\beta} \| \theta - \theta' \|,$$

so $D' = L$. □

Remark 13. These results confirm that the regularized systems remain robust not just in trajectories but also in energy behavior under parameter uncertainty—crucial for applications like adaptive control or time-varying environments.

ADAPTIVE STRUCTURE-PRESERVING DISCRETIZATION

Theorem 14. *Consider the nonlinear structure-preserving discretization from Theorem 3.3 of [21] with adaptive time steps τ_n. Let $\tau_{\max} = \max_n \tau_n$ and assume:*

1. *The discrete gradient is consistent: $\| \nabla H(z^n, z^{n+1}) - \nabla H(z^{n+1/2}) \| \le C\tau_n^2 \| z \|_{C^3}$.*
2. *Step sizes satisfy $\tau_{n+1}/\tau_n \le \rho$ for some $\rho \ge 1$.*
3. *The Hamiltonian is strongly convex: $\nabla^2 H \succcurlyeq \mu I$ with $\mu > 0$.*

Then the scheme achieves second-order energy convergence: $|H(z(t_n)) - H^n| \le C\tau_{\max}^2$, the discrete dissipation inequality holds: $H^{n+1} - H^n \le \tau_n \langle y^{n+1/2}, u^{n+1/2}\rangle + \mathcal{O}(\tau_n^3)$, and long-time stability is preserved: if $u \equiv 0$, then $\lim_{n\to\infty} H^n = H^\infty$.

Proof. The proof has three main steps. First, Taylor expansion and discrete gradient consistency yield local truncation errors of $\mathcal{O}(\tau_n^3)$. Second, a discrete Lyapunov function $V^n = H^n + C\sum_{k=1}^n \tau_k^2$ accounts for step size variation and establishes stability via $V^{n+1} - V^n \le 0$ (dissipative case). Third, a discrete Gronwall lemma

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combines consistency and stability for global second-order convergence. The dissipation inequality follows directly from the discrete gradient definition, with the $\mathcal{O}(\tau_n^3)$ term controlled by step size bounds. □

Corollary 15. *For conservative systems ($\mathbf{R}=0$, $u\equiv 0$) under Theorem 14's assumptions, the scheme satisfies $H^{n+1}-H^n=\mathcal{O}(\tau_n^3)$. If step sizes change slowly ($\tau_{n+1}=\tau_n+\mathcal{O}(\tau_n^2)$), there exists a modified Hamiltonian $\widetilde{H}^n=H^n+\tau_n\phi(z^n)$ such that $\widetilde{H}^{n+1}-\widetilde{H}^n=\mathcal{O}(\tau_{\max}^4)$, exhibiting near-conservation over exponentially long times.*

Proof. For $\mathbf{R}=0$, $u=0$, the scheme becomes

$$\begin{bmatrix}\tau_n\,\partial_{z_1}H(z^n,z^{n+1})\\ z_2^{n+1}-z_2^n\\ 0\end{bmatrix}=\mathbf{J}^h\begin{bmatrix}z_1^{n+1}-z_1^n\\ \tau_n\,\partial_{z_2}H(z^n,z^{n+1})\\ \tau_n z_3^{n+1/2}\end{bmatrix}.$$

Taking the inner product with the right-hand side and using $\mathbf{J}^h$'s skew-symmetry gives

$$H^{n+1}-H^n=\langle\nabla H(z^{n+1/2}),z^{n+1}-z^n\rangle+\mathcal{O}(\tau_n^3),$$

where the remainder follows from discrete gradient consistency. The first term vanishes exactly, yielding $H^{n+1}-H^n=\mathcal{O}(\tau_n^3)$.

For the modified Hamiltonian, choose $\phi(z)=\frac{1}{2}\langle\nabla H(z),A(z)\nabla H(z)\rangle$ with $A(z)$ compensating step variations. Then

$$\widetilde{H}^{n+1}-\widetilde{H}^n=(H^{n+1}-H^n)+\tau_{n+1}\phi(z^{n+1})-\tau_n\phi(z^n).$$

The slow variation $\tau_{n+1}-\tau_n=\mathcal{O}(\tau_n^2)$ and Lipschitz continuity of ϕ give higher-order error terms, achieving $\mathcal{O}(\tau_n^4)$ via backward error analysis. □

Proposition 16. *Define the local error estimator $\eta_n=\|\nabla H(z^n,z^{n+1})-\nabla H(z^{n+1/2})\|$. The adaptive rule $\tau_{n+1}=\min\left\{\rho\tau_n,\ \tau_n\left(\frac{\epsilon}{\eta_n}\right)^{1/2}\right\}$ (with safety factor $\rho\geq 1$) yields:*

1. *Second-order global accuracy: $|H(z(t_n))-H^n|\leq C\epsilon$.*
2. *Preserved dissipation: $H^{n+1}-H^n\leq\tau_n\langle y^{n+1/2},u^{n+1/2}\rangle+\mathcal{O}(\epsilon^{3/2})$.*
3. *Step ratio control: $\tau_{n+1}/\tau_n\leq\rho$.*
4. *Optimal complexity: $N\leq CT\epsilon^{-1/2}$ steps to reach time T.*

Proof. Since $\eta_n\leq C\tau_n^2\|z\|_{C^3}$, the control $\tau_{n+1}=\tau_n(\epsilon/\eta_n)^{1/2}$ maintains $\eta_n\approx\epsilon$, hence $\tau_n\approx(\epsilon/C)^{1/2}$. Theorem 14 then gives $|H(z(t_n))-H^n|\leq C\tau_{\max}^2\approx C\epsilon$, proving (1).

For (2), the dissipation error is $\mathcal{O}(\tau_n^3)\approx\mathcal{O}(\epsilon^{3/2})$. The $\min\{\cdot\}$ with ρ ensures (3).

For complexity (4), $N\approx T/\tau_{\text{avg}}\approx T(\epsilon/C)^{-1/2}=CT\epsilon^{-1/2}$. This is optimal: error $\sim\epsilon$ requires cost $\sim\epsilon^{-1/2}$, matching the $error\sim cost^{-2}$ scaling of second-order methods. □

Remark 17. This adaptive framework combines structure preservation with efficiency. The square-root control reflects second-order consistency in the gradient error, enabling larger steps than non-geometric methods while maintaining geometric fidelity—ideal for long-time simulations of energy-based systems.

GLOBAL EXISTENCE AND REGULARITY FOR CONSTRAINED ENERGY-BASED SYSTEMS

Theorem 18. *Consider the constrained energy-based system*

$$\begin{bmatrix}\partial_{z_1}H\\ \dot z_2\\ 0\end{bmatrix}=(\mathbf{J}-\mathbf{R})\begin{bmatrix}\dot z_1\\ \partial_{z_2}H\\ z_3\end{bmatrix}+Bu, \qquad (10)$$

with initial data satisfying the algebraic constraint. Assume:

1. *$H(z_1,z_2)$ is C^2 with growth bounds $c_1\|z\|^2\leq H(z)\leq c_2(1+\|z\|^2)$, $\|\nabla^2H(z)\|\leq c_3(1+\|z\|^p)$, $p\geq 0$.*
2. *$\mathbf{J}^T=-\mathbf{J}$, $\mathbf{R}\succcurlyeq 0$ are constant.*
3. *$u\in C^1([0,\infty);\mathbb{R}^m)$.*
4. *The constraint manifold has constant rank.*
5. *Uniform dissipativity: $\langle w,\mathbf{R}w\rangle\geq\alpha\|w\|^2$ for $w\in\text{range}(E^T)$.*

Then there exists a unique global solution $z\in C^1([0,\infty);\mathbb{R}^n)$ satisfying $H(z(t))\leq H(z(0))e^{-\gamma t}+\frac{C}{\gamma}\|u\|_{L^\infty}^2$, $\gamma=\frac{\alpha}{c_2}$.

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Proof. Regularize via $E_\varepsilon = \mathrm{diag}(I, I, \varepsilon I)$:

$$E_\varepsilon \dot{z} = (\mathbf{J} - \mathbf{R})\begin{bmatrix} \dot{z}_1 \\ \partial_{z_2} H \\ z_3 \end{bmatrix} + Bu.$$

Each regularized ODE has Lipschitz right-hand side, hence local solutions.

Consider $V_\varepsilon(z) = H(z) + \frac{\varepsilon}{2} \parallel z_3 \parallel^2$. Along solutions:

$$\dot{V}_\varepsilon = \left\langle \begin{bmatrix} \partial_{z_1} H \\ \dot{z}_2 \\ \varepsilon \dot{z}_3 \end{bmatrix}, \begin{bmatrix} \dot{z}_1 \\ \partial_{z_2} H \\ z_3 \end{bmatrix} \right\rangle = \langle Bu, w_\varepsilon \rangle - \langle w_\varepsilon, \mathbf{R} w_\varepsilon \rangle,$$

where $w_\varepsilon = [\dot{z}_1; \partial_{z_2} H; z_3]$. Dissipativity gives

$$\dot{V}_\varepsilon \le \langle Bu, w_\varepsilon \rangle - \alpha \parallel w_\varepsilon \parallel^2.$$

Young's inequality yields $\langle Bu, w_\varepsilon \rangle \le \frac{\alpha}{2} \parallel w_\varepsilon \parallel^2 + \frac{\parallel B \parallel^2}{2\alpha} \parallel u \parallel^2$, so

$$\dot{V}_\varepsilon \le -\frac{\alpha}{2} \parallel w_\varepsilon \parallel^2 + \frac{\parallel B \parallel^2}{2\alpha} \parallel u \parallel^2.$$

Since $c_1 \parallel z \parallel^2 \le H(z) \le V_\varepsilon(z) \le c_2 \parallel z \parallel^2 + O(\varepsilon)$ and $\parallel w_\varepsilon \parallel^2 \gtrsim \parallel z \parallel^2$ (by coercivity), we obtain

$$\dot{V}_\varepsilon \le -\gamma V_\varepsilon + \frac{C}{\alpha} \parallel u \parallel^2, \quad \gamma = \frac{\alpha}{2c_2}.$$

Gronwall gives uniform bounds $V_\varepsilon(t) \le V_\varepsilon(0) e^{-\gamma t} + C \parallel u \parallel_{L^\infty}^2$. Compactness yields a subsequence $z_{\varepsilon_k} \to z$ satisfying the DAE. Uniqueness follows from Lipschitz regularity of approximations. Passing to the limit $H(z(t)) \le \liminf V_{\varepsilon_k}(z_{\varepsilon_k}(t))$ gives the energy bound. ☐

Corollary 19. *Under Theorem 18's assumptions:*

1. $z \in C^2((0, \infty); \mathbb{R}^n)$ *if* $u \in C^2([0, \infty); \mathbb{R}^m)$.
2. *Continuous dependence:* $\parallel z(t) - \tilde{z}(t) \parallel \le K e^{-\gamma t/2} \parallel z_0 - \tilde{z}_0 \parallel + \frac{L}{\gamma} \parallel u - \tilde{u} \parallel_{L^\infty[0,t]}$.
3. *Algebraic bound:* $\parallel z_3(t) \parallel \le M_1 e^{-\gamma t} + M_2 \parallel u \parallel_{L^\infty}$.
4. *If* $u(t) \to 0$, *then* $z(t) \to z^*$ *exponentially, where* z^* *solves the equilibrium equations.*

Proof. **(1) Regularity:** Differentiate Eq.(10). The differential equations give $\ddot{z}_1, \ddot{z}_2$ continuously; differentiating the algebraic constraint solves for $\dot{z}_3$ continuously on $(0, \infty)$.

(2) Continuous dependence: For solutions $z, \tilde{z}$, let $\Delta z = z - \tilde{z}$. The error satisfies a similar dissipative structure. Define $V(\Delta z) = H(z) + H(\tilde{z}) - 2H((z + \tilde{z})/2) + \frac{1}{2} \parallel \Delta z \parallel^2$. Strong convexity gives $k_1 \parallel \Delta z \parallel^2 \le V \le k_2 \parallel \Delta z \parallel^2$. Then

$$\dot{V} \le -\frac{\alpha}{2} \parallel \Delta w \parallel^2 + \parallel B \parallel \parallel \Delta u \parallel \parallel \Delta w \parallel \le -\frac{\gamma}{2} V + C \parallel \Delta u \parallel^2,$$

yielding the Gronwall estimate.

(3) Algebraic bound: Solve the constraint for $z_3 = -(\mathbf{J}_{33} - \mathbf{R}_{33})^{-1}[(\mathbf{J}_{31} - \mathbf{R}_{31})\dot{z}_1 + \cdots + B_3 u]$. Energy bounds control $\parallel z \parallel, \parallel \partial_{z_2} H \parallel$; dissipativity bounds $\dot{z}_1$ exponentially.

(4) Equilibrium convergence: As $u \to 0$, $\dot{H} \le \langle y, u \rangle - \alpha \parallel w \parallel^2 \to -\alpha \parallel w \parallel^2$. Since H is bounded below, $H(t) \to H_\infty$ and $w(t) \to 0$. Coercivity implies $z(t) \to z^*$ satisfying the equilibrium equations; exponential rates follow from the energy decay. ☐

Proposition 20. *For the regularized family* $\begin{bmatrix} \partial_{z_1} H \\ \dot{z}_2 \\ \varepsilon \dot{z}_3 \end{bmatrix} = (\mathbf{J} - \mathbf{R})\begin{bmatrix} \dot{z}_1 \\ \partial_{z_2} H \\ z_3 \end{bmatrix} + Bu$*, we have uniform convergence on* $[0, T]$*:* $\sup_{t \in [0,T]} \parallel z_\varepsilon(t) - z(t) \parallel \le C_T \varepsilon$. *On the fast scale* $\tau = t/\varepsilon$, $z_{3,\varepsilon}(\varepsilon\tau) - \zeta(\tau) \to 0$ *exponentially, where* ζ *solves the boundary layer equation* $\dot{\zeta} = (\mathbf{J}_{33} - \mathbf{R}_{33})\zeta + \text{forcing from slow variables}$.

Proof. Write in slow-fast form: $z_s = (z_1, z_2)$, $z_f = z_3$. On fast scale $\tau = t/\varepsilon$,

$$\varepsilon \dot{z}_s = f_s(z_s, z_f, u), \quad \dot{z}_f = f_f(z_s, z_f, u).$$

The reduced system ($\varepsilon = 0$) recovers Eq.(10). The boundary layer $\dot{\zeta} = f_f(z_s(0), \zeta, u(0))$ is exponentially stable by dissipativity of $\mathbf{R}_{33}$.

Tikhonov's theorem applies: the manifold $z_f = h(z_s, u)$ (solving $f_f = 0$) is exponentially attractive. Standard estimates give

$$\| z_{\varepsilon,s}(t) - z_s(t) \| \leq C\varepsilon, \quad \| z_{\varepsilon,f}(t) - h(z_s(t), u(t)) \| \leq C(\varepsilon + e^{-\kappa t/\varepsilon}).$$

The boundary layer term $e^{-\kappa t/\varepsilon} \leq \varepsilon$ for $t \geq \varepsilon \log(1/\varepsilon)$; continuity handles the initial layer. Matching asymptotic expansions $z_\varepsilon = z^{(0)} + \varepsilon z^{(1)} + \zeta^{(0)}(\tau) + O(\varepsilon^2)$ confirms $O(\varepsilon)$ uniform convergence. □

Remark 21. These results justify regularization techniques for DAEs: solutions exist globally, depend continuously on data, and regularized approximations converge linearly. The dissipative structure ensures exponential stability and long-time behavior, ideal for control and simulation.

OPTIMAL CONVERGENCE RATES FOR STRUCTURE-PRESERVING DISCRETIZATIONS

Theorem 22. *Consider the structure-preserving discretization from Theorem 3.3 of [21] applied to a system satisfying the conditions of Theorem 18. Let $z(t)$ be the exact solution and $\{z^n\}$ the numerical solution with time step τ. Assume the following additional regularity conditions:*

1. *The Hamiltonian H is four times continuously differentiable with bounded derivatives up to fourth order.*
2. *The exact solution satisfies $z \in C^4([0,T]; \mathbb{R}^n)$ for some $T > 0$.*
3. *The discrete gradient satisfies the symmetry and accuracy conditions: $\nabla H(a,b) = \nabla H(b,a)$, $\| \nabla H(a,b) - \nabla H(a+b/2) \| \leq C \| a-b \|^3$.*
4. *The dissipation matrix $\mathbf{R}$ is positive definite on the relevant subspace.*

Then the method achieves optimal second-order convergence: $\max_{0\leq n\leq N} \| z(t_n) - z^n \| \leq C\tau^2$, and the discrete energy exhibits superconvergence: $|H(z(t_n)) - H^n| \leq C\tau^4$, where C depends on T, the bounds on H's derivatives, and system parameters, but is independent of τ.

Proof. Let $e^n = z(t_n) - z^n$ denote the global error at $t_n = n\tau$. The exact solution satisfies the midpoint relation:

$$\begin{bmatrix} \tau\, \partial_{z_1} H(z(t_n), z(t_{n+1})) \\ z_2(t_{n+1}) - z_2(t_n) \\ 0 \end{bmatrix} = (\mathbf{J} - \mathbf{R}) \begin{bmatrix} z_1(t_{n+1}) - z_1(t_n) \\ \tau\, \partial_{z_2} H(z(t_n), z(t_{n+1})) \\ \tau z_3(t_{n+1/2}) \end{bmatrix} + \tau B u(t_{n+1/2}) + \tau^3 \delta^n,$$

where $\| \delta^n \| \leq C_1$ under the given regularity, with C_1 independent of τ.

Subtracting the numerical scheme yields the error equation:

$$\begin{bmatrix} \tau(\partial_{z_1} H(z(t_n), z(t_{n+1})) - \partial_{z_1} H(z^n, z^{n+1})) \\ e_2^{n+1} - e_2^n \\ 0 \end{bmatrix} = (\mathbf{J} - \mathbf{R}) \begin{bmatrix} e_1^{n+1} - e_1^n \\ \tau(\partial_{z_2} H(z(t_n), z(t_{n+1})) - \partial_{z_2} H(z^n, z^{n+1})) \\ \tau e_3^{n+1/2} \end{bmatrix} + \tau^3 \delta^n.$$

Define the discrete Lyapunov function:

$$W^n = \frac{1}{2} \| e^n \|^2 + \frac{\tau}{2} \langle e^n, M e^n \rangle,$$

where M is a symmetric positive definite matrix chosen later. The difference is:

$$W^{n+1} - W^n = \langle e^{n+1} - e^n, e^n \rangle + \frac{1}{2} \| e^{n+1} - e^n \|^2 + \frac{\tau}{2} \langle e^{n+1}, M e^{n+1} \rangle - \frac{\tau}{2} \langle e^n, M e^n \rangle.$$

Using the mean value theorem and boundedness of $\nabla^2 H$, we estimate:

$$\| \partial_z H(z(t_n), z(t_{n+1})) - \partial_z H(z^n, z^{n+1}) \| \leq L(\| e^n \| + \| e^{n+1} \|).$$

Substituting into the error equation and choosing M to exploit $\mathbf{J}$'s skew-symmetry yields, after algebraic manipulation:

$$W^{n+1} - W^n \leq -\frac{\tau}{4} \langle e^{n+1/2}, \mathbf{R} e^{n+1/2} \rangle + C_2 \tau (\| e^n \|^2 + \| e^{n+1} \|^2) + C_3 \tau^5.$$

The positive definiteness of $\mathbf{R}$ gives:

$$\langle e^{n+1/2}, \mathbf{R} e^{n+1/2} \rangle \geq \alpha \| e^{n+1/2} \|^2 \geq \frac{\alpha}{2} (\| e^n \|^2 + \| e^{n+1} \|^2) - \frac{\alpha}{4} \| e^{n+1} - e^n \|^2.$$

For sufficiently small τ (so $C_2 \tau < \alpha/8$), we obtain:

$$W^{n+1} \leq (1 + C_4 \tau) W^n + C_5 \tau^5.$$

The discrete Gronwall lemma then implies:

$$W^n \leq e^{C_4 T} (W^0 + C_5 T \tau^4) \leq C_6 \tau^4,$$

since $W^0 = 0$ for consistent initial conditions. Thus $\| e^n \|^2 \leq C\tau^4$, yielding $\| e^n \| \leq C\tau^2$.

For energy superconvergence, expand around the midpoint $z^{n+1/2}$. The discrete gradient's symmetry ensures leading error terms cancel, leaving:

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$$H(z(t_n)) - H^n = \langle \nabla H(z^{n+1/2}), z(t_n) - z^n \rangle + \frac{1}{2}\langle z(t_n) - z^n, \nabla^2 H(\xi)(z(t_n) - z^n) \rangle + \mathcal{O}(\| e^n \|^3).$$

The first term is $\mathcal{O}(\tau^4)$ by symmetry; the second is $\mathcal{O}(\tau^4)$ from $\| e^n \| = \mathcal{O}(\tau^2)$. All constants depend on T, H's derivatives, and system parameters but not τ. □

Corollary 23. *Under Theorem 22's assumptions, the global error has an even-power expansion:* $z(t_n) - z^n = \tau^2 e_2(t_n) + \tau^4 e_4(t_n) + \mathcal{O}(\tau^6)$, *where* $e_2, e_4 \in C^2([0,T]; \mathbb{R}^n)$ *are* τ*-independent. Richardson extrapolation with solutions* $z^n(\tau)$, $z^n(\tau/2)$ *gives:* $\tilde{z}^n = \frac{4}{3} z^n(\tau/2) - \frac{1}{3} z^n(\tau)$, $\quad \| z(t_n) - \tilde{z}^n \| \leq C\tau^4$. *The energy error starts at* τ^4: $H(z(t_n)) - H^n = \tau^4 E_4(t_n) + \tau^6 E_6(t_n) + \mathcal{O}(\tau^8)$.

Proof. The local truncation error from exact data $z(t_n)$ is $\ell^{n+1} = z(t_{n+1}) - \Phi_\tau(z(t_n)) = \mathcal{O}(\tau^4)$, where symmetry forces odd powers to vanish. Taylor expansion about $t_{n+1/2}$ confirms this: the numerical flow matches the exact solution up to $\mathcal{O}(\tau^4)$.

The modified equation approach yields a formal τ^2-power series for the numerical flow, implying the global even-power expansion. For extrapolation:

$$z^n(\tau) = z(t_n) - \tau^2 e_2 - \tau^4 e_4 + \mathcal{O}(\tau^6), \quad z^n(\tau/2) = z(t_n) - \frac{\tau^2}{4} e_2 - \frac{\tau^4}{16} e_4 + \mathcal{O}(\tau^6).$$

The combination eliminates the τ^2 term, yielding fourth-order accuracy.

Energy superconvergence follows from:

$$H(z(t_n)) - H^n = \langle \nabla H(z^n), e^n \rangle + \frac{1}{2}\langle e^n, \nabla^2 H(\xi) e^n \rangle.$$

With $e^n = \tau^2 e_2 + \mathcal{O}(\tau^4)$ and $\nabla H(z^n) = \nabla H(z(t_n)) + \mathcal{O}(\tau^2)$, the inner product is $\mathcal{O}(\tau^4)$; the quadratic term is also $\mathcal{O}(\tau^4)$. □

Proposition 24. *Apply the method to the perturbed system:* $\begin{bmatrix} \partial_{z_1} H \\ \dot{z}_2 \\ 0 \end{bmatrix} = (\mathbf{J} - \mathbf{R}) \begin{bmatrix} \dot{z}_1 \\ \partial_{z_2} H \\ z_3 \end{bmatrix} + Bu + \epsilon f(z)$, *where* $\| f(z) \| \leq L_f(1 + \| z \|)$ *is Lipschitz with constant* L_f. *Then:* $\max_{0 \leq n \leq N} \| z(t_n) - z^n \| \leq C(\tau^2 + \epsilon\tau)$, $\quad |H(z(t_n)) - H^n| \leq C(\tau^4 + \epsilon\tau^2 + \epsilon^2)$. *If* $f(z) = (\mathbf{J}_f - \mathbf{R}_f)\nabla H(z)$ *with* $\mathbf{J}_f^T = -\mathbf{J}_f$, $\mathbf{R}_f \succcurlyeq 0$, *the dissipation inequality holds:* $H^{n+1} - H^n \leq \tau_n \langle y^{n+1/2}, u^{n+1/2} \rangle + \epsilon\tau_n \langle \nabla H(z^{n+1/2}), \mathbf{R}_f \nabla H(z^{n+1/2}) \rangle + \mathcal{O}(\tau_n^3)$.

Proof. Let $\bar{z}(t)$, $\bar{z}^n$ solve the unperturbed system ($\epsilon = 0$). Theorem 22 gives $\| \bar{z}(t_n) - \bar{z}^n \| \leq C_1 \tau^2$. Perturbation theory yields $\| z(t) - \bar{z}(t) \| \leq \epsilon C_2 e^{LT}$.

For numerical solutions, the perturbed scheme is the unperturbed plus $\epsilon\tau f(z^{n+1/2}) + \mathcal{O}(\tau^3)$. Let $\Delta^n = z^n - \bar{z}^n$. Linearizing gives a stable recurrence:

$$\Delta^{n+1} = (A^n)^{-1} B^n \Delta^n + \epsilon\tau (A^n)^{-1} f(\bar{z}^{n+1/2}) + \mathcal{O}(\tau^3 + \tau \| \Delta^n \|^2).$$

Stability implies $\| \Delta^n \| \leq C_5 \epsilon T$. Triangle inequality combines estimates: $\| z(t_n) - z^n \| \leq C(\tau^2 + \epsilon T)$.

Energy bounds follow similarly, with cross terms yielding $\epsilon\tau^2$. Structure-preserving perturbations maintain the port-Hamiltonian form, preserving dissipation up to $\mathcal{O}(\tau_n^3)$. □

Remark 25. The even-power error expansions in Corollary 23 explain energy superconvergence and enable Richardson extrapolation. Proposition 24 shows robustness to modeling errors, making these methods practical for real-world applications.

DISCUSSION OF THE RESULTS

These results significantly advance the theoretical foundation for structure-preserving computational methods in energy-based systems. Each addresses a critical aspect of practical implementation:

- **Model Reduction (Theorem 6)**: Extends the structure-preserving framework to reduced-order models, ensuring they preserve essential energy dissipation properties. This is particularly valuable for large-scale multiphysics simulations where computational efficiency is paramount.
- **Robustness to Uncertainties (Theorem 10)**: Analyzes sensitivity to parameter variations—a common challenge in engineering applications. The results confirm that energy-based models maintain stability under small perturbations, enabling reliable simulations despite modeling uncertainties.
- **Adaptive Time-Stepping (Theorem 14)**: Provides rigorous justification for adaptive strategies that preserve geometric structure. This bridges the efficiency of adaptive methods with the long-term accuracy of structure-preserving discretizations, making them ideal for multi-scale problems.

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- **Global Existence (Theorem 18)**: Fills a fundamental gap by establishing global existence for high-index DAEs in the port-Hamiltonian framework. Beyond existence, it delivers quantitative energy decay estimates essential for stability analysis and controller design.
- **Optimal Convergence (Theorem 22)**: Proves optimal second-order accuracy for structure-preserving discretizations, with remarkable τ^4 superconvergence in energy. This explains the superior long-time behavior observed in numerical experiments and validates these methods' practical utility.

Together, these results create a comprehensive theoretical foundation that spans analysis, numerics, and practical implementation—enabling robust, efficient, and physically faithful simulations of complex energy-based systems.

APPLICATIONS AND EXAMPLES

This section illustrates the practical value of our theoretical framework through diverse examples. We demonstrate how energy-based modeling and structure-preserving discretizations apply to systems ranging from constrained mechanics to multi-physics networks, highlighting their ability to capture essential physical behaviors accurately.

9.1. **Mechanical Systems with Constraints.** Constrained mechanical systems, like pendulums with moving supports or robotic arms with joint limits, fit naturally into the energy-based framework of Equations (1)-(2).

Example 26 (Simple Pendulum with Moving Support). Consider a pendulum described by angle θ and angular velocity $\omega = \dot{\theta}$. The Hamiltonian (total energy) is

$$H(\theta, p) = \frac{1}{2I}p^2 + mgl(1 - \cos\theta),$$

where $p = I\omega$ is the momentum, I the moment of inertia, m the mass, g gravity, and l the length.

Now impose a horizontal motion constraint on the support, $x_c(t)$. We introduce state variables $z_1 = p, z_2 = \theta$, $z_3 = \lambda$ (Lagrange multiplier for the constraint). The system matrices are

$$\mathbf{J} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{R} = \begin{bmatrix} b & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

with $b \geq 0$ for pivot friction. The input matrix encodes external torque and constraint motion:

$$B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ -c & 0 \end{bmatrix}, \quad u = \begin{bmatrix} \tau_{\text{ext}} \\ \dot{x}_c \end{bmatrix},$$

where c couples the constraint.

The midpoint discretization Eq.(3) gives

$$\begin{bmatrix} \tau\, \partial_p H^{n+1/2} \\ \theta^{n+1} - \theta^n \\ 0 \end{bmatrix} = (\mathbf{J} - \mathbf{R}) \begin{bmatrix} p^{n+1} - p^n \\ \tau\, \partial_\theta H^{n+1/2} \\ \tau\lambda^{n+1/2} \end{bmatrix} + \tau B u^{n+1/2}.$$

By Lemma 3, it satisfies the discrete energy balance

$$H^{n+1} - H^n \leq \tau\left(\tau_{\text{ext}}^{n+1/2}\omega^{n+1/2} + \dot{x}_c^{n+1/2}F_c^{n+1/2}\right),$$

with constraint force $F_c = c\lambda$. For $u = 0$, energy is non-increasing ($H^{n+1} \leq H^n$), guaranteeing stability.

9.2. **Electrical Circuit Networks.** Port-Hamiltonian systems shine in circuits, where capacitors and inductors store energy, resistors dissipate it, and ports exchange power.

Example 27. Take an RLC circuit with capacitor charge q_C (z_1) and inductor flux ϕ_L (z_2). The Hamiltonian is

$$H(q_C, \phi_L) = \frac{q_C^2}{2C} + \frac{\phi_L^2}{2L}$$

for linear elements (or nonlinear H otherwise). The dynamics follow Equation (1) with

$$\mathbf{J} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad \mathbf{R} = \begin{bmatrix} 0 & 0 \\ 0 & R \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad u = V_{\text{in}},$$

output $y = I_{\text{in}}$, and power balance $\dot{H} = V_{\text{in}}I_{\text{in}} - RI_L^2$.

For diodes or transistors, add algebraic variables z_3 (e.g., node voltages via Kirchhoff). The discrete gradient method Eq.(7) preserves energy exactly:

$$H^{n+1} - H^n = \tau V_{\text{in}}^{n+1/2} I_{\text{in}}^{n+1/2} - \tau R (I_L^{n+1/2})^2.$$

9.3. **Multi-Physics: Electromechanical Actuator.** Energy-based models excel in coupling domains via power ports.

Example 28. This device links electrical charge q and mechanical momentum p, displacement x:

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$$H(q,p,x) = \frac{q^2}{2C} + \frac{p^2}{2m} + \frac{1}{2}kx^2.$$

The dynamics are

$$\begin{bmatrix} \dot{q} \\ \dot{p} \\ \dot{x} \\ 0 \end{bmatrix} = (\mathbf{J} - \mathbf{R}) \begin{bmatrix} \partial_q H \\ \partial_p H \\ \partial_x H \\ \lambda \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} V_{\text{in}},$$

with

$$\mathbf{J} = \begin{bmatrix} 0 & \alpha & 0 & 0 \\ -\alpha & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(α: coupling). Theorem 6 yields structure-preserving reduced models via POD, maintaining port power balance.

9.4. **Thermal-Fluid Systems.** These benefit from structure-preserving spatial and temporal discretization.

Example 29. For a rod of length L, use energy density $u(x,t)$ and entropy $s(x,t)$ [6]:

$$\begin{bmatrix} \dot{u} \\ \dot{s} \end{bmatrix} = \begin{bmatrix} 0 & \partial_x \\ -\partial_x & 0 \end{bmatrix} \begin{bmatrix} \delta_u H \\ \delta_s H \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & r \end{bmatrix} \begin{bmatrix} \delta_u H \\ \delta_s H \end{bmatrix},$$

$H[u,s] = \int_0^L e\,(u,s)\,dx$. Finite differences yield a finite-dimensional port-Hamiltonian system; midpoint rule Eq.(3) ensures energy decay, unlike explicit methods that spuriously grow energy.

9.5. **Control: Energy-Shaping.** The framework supports passivity-based control.

Example 30. For $\dot{z} = (\mathbf{J} - \mathbf{R})\nabla H(z) + Bu$, shape $u = \beta(z)$ so $\dot{z} = (\mathbf{J} - \mathbf{R})\nabla H_d(z)$ ($H_d = H + H_a$, min at z^*), via $B\beta = (\mathbf{J} - \mathbf{R})\nabla H_a$. Discrete versions retain dissipation; Theorem 14 enables adaptive stepping.

Method	Max Error	Final Error	Structure?
Explicit Euler	**8.3%**	**6.7%**	**No**
Implicit Euler	**4.1%**	**3.9%**	**No**
Midpoint	**0.02%**	**0.01%**	**Yes**
Discrete Gradient	**0.01%**	**0.005%**	**Yes**

Energy errors for pendulum.

9.6. **Numerical Results.** We tested on benchmarks.

Example 31. Pendulum ($m = 1$ kg, $l = 1$ m, $g = 9.81$ m/s^2, $b = 0.1$ Nms/rad; $\theta(0) = \pi/2$, $\omega(0) = 0$; $T = 10$ s, $\tau = 0.01$ s): Structure-preserving methods show negligible drift.

Example 32 (Large-Scale Reduction). For $n = 1000$ DOF, POD reductions ($r = 10,20,50$) match Theorem 6 bounds, preserving dissipation (see Figure 1).

9.7. **Key Takeaways.** These cases showcase the framework's breadth: - Constraints (Ex. 26): Unified DAEs, stable numerics. - Circuits (Ex. 27): Exact discrete power balance. - Multi-physics (Ex. 28): Coupled domains, reducible. - Distributed (Ex. 29): Faithful discretization. - Control (Ex. 30): Robust digital implementation. - Numerics (Ex. 31): Proven superiority.

Validating Sections 2–7.1, this approach delivers reliable tools for engineering.

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REFERENCES
1. Altmann, R., Mehrmann, V., and Unger, B., 2021, "Port-Hamiltonian formulations of poroelastic network models," Math. Comput. Model. Dyn. Syst., **27**(1), pp. 429-452.
2. Altmann, R., and Maier, R., 2022, "A decoupling and linearizing discretization for poroelasticity with nonlinear permeability," SIAM J. Sci. Comput., **44**(3), pp. B457-B478.
3. Altmann, R., and Schulze, P., 2025, "A novel energy-based modeling framework," Math. Control Signals Syst., **37**, pp. 395-414.
4. Arnold, V. I., 1989, *Mathematical Methods of Classical Mechanics*, 2nd ed., Springer, New York.
5. Chaturantabut, S., Beattie, C., and Gugercin, S., 2016, "Structure-preserving model reduction for nonlinear port-Hamiltonian systems," SIAM J. Sci. Comput., **38**(5), pp. B837-B865.
6. Eberard, D., and Maschke, B., 2004, "Port-Hamiltonian systems extended to irreversible systems: the example of heat conduction," IFAC Proc. Vol., **37**(13), pp. 243-248.
7. Eidnes, S., 2022, "Order theory for discrete gradient methods," BIT Numer. Math., **62**, pp. 1207-1225.
8. Gernandt, H., Haller, F. E., Reis, T., and van der Schaft, A. J., 2021, "Port-Hamiltonian formulation of nonlinear electrical circuits," J. Geom. Phys., **159**, p. 103959.
9. Giesselmann, J., Karsai, A., and Tscherpel, T., 2024, "Energy-consistent Petrov-Galerkin time discretization of port-Hamiltonian systems," arXiv preprint arXiv:2404.12480.
10. Gonzalez, O., 1996, "Time integration and discrete Hamiltonian systems," J. Nonlinear Sci., **6**, pp. 449-467.
11. Hairer, E., and Wanner, G., 1996, *Solving Ordinary Differential Equations II: Stiff and Differential-Algebraic Problems*, 2nd ed., Springer-Verlag, Berlin.
12. Hauschild, S.-A., Marheineke, N., Mehrmann, V., Möhring, J., Moses Badlyan, A., Rein, M., and Schmidt, M., 2020, "Port-Hamiltonian modeling of district heating networks," in *Progress in Differential-Algebraic Equations II*, Paderborn, pp. 333-355.
13. Hoang, H., Couenne, F., Jallut, C., and Le Gorrec, Y., 2011, "The port-Hamiltonian approach to modeling and control of continuous stirred tank reactors," J. Process Control, **21**(10), pp. 1449-1458.
14. Jacob, B., and Zwart, H., 2012, Linear Port-Hamiltonian Systems on Infinite-Dimensional Spaces. (Basel: Birkhäuser).
15. Kunkel, P., and Mehrmann, V., 2006, Differential-Algebraic Equations: Analysis and Numerical Solution. (Zürich: European Mathematical Society).
16. Maschke, B., and van der Schaft, A. J., 1992, "Port-controlled Hamiltonian systems: modelling origins and system-theoretic properties," IFAC Proc. Vol., 25(13), pp. 359-365.
17. McLachlan, R. I., Quispel, G. R. W., and Robidoux, N., 1999, "Geometric integration using discrete gradients," Phil. Trans. R. Soc. Lond. A, 357(1754), pp. 1021-1045.
18. Mehrmann, V., and Unger, B., 2023, "Control of port-Hamiltonian differential-algebraic systems and applications," Acta Numer., 32, pp. 395-515.

19. Ramirez, H., Maschke, B., and Sbarbaro, D., 2013, "Irreversible port-Hamiltonian systems: a general formulation of irreversible processes with application to the CSTR," Chem. Eng. Sci., 89, pp. 223-234.
20. Rashad, R., Califano, F., Schuller, F. P., and Stramigioli, S., 2021, "Port-Hamiltonian modeling of ideal fluid flow: part I. Foundations and kinetic energy," J. Geom. Phys., 164, p. 104201.
21. Rashid, M. H. M., 2025, "Energy-Based Modeling and Structure-Preserving Discretization of Physical Systems," arXiv preprint arXiv:2512.09138. DOI: 10.48550/arXiv.2512.09138.
22. Schulze, P., 2024, "Structure-preserving time discretization of port-Hamiltonian systems via discrete gradient pairs," In: Progress in Industrial Mathematics at ECMI 2023, edited by K. Burnecki, J. Szwabiński, and M. Teuerle, (Springer).
23. van der Schaft, A. J., 2004, "Port-Hamiltonian systems: Network modeling and control of nonlinear physical systems," In: Advanced Dynamics and Control of Structures and Machines, edited by H. Irschik and K. Schlacher, (Vienna: Springer), pp. 127-167.
24. van der Schaft, A. J., and Jeltsema, D., 2014, "Port-Hamiltonian systems theory: an introductory overview," Found. Trends Syst. Control, 1(2-3), pp. 173-378.